Computing Permanents of Complex Diagonally Dominant Matrices and Tensors

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ABSTRACT: We prove that for any > 1, xed in advance, the permanent of an n n complex matrix, where the absolute value of each diagonal entry is at leasttimes bigger than the sum of the absolute values of all other entries in the same row, can be approximated within any relative error $0 \ll 1$ in quasi-polynomial $n^{O(\ln\ln)}$ time. We extend this result to multidimensional permanents of tensors and apply it to weighted counting of perfect matchings in hypergraphs.

INTRODUCTION AND MAIN RESULTS

In this paper, we continue the line of research started in [Ba16a] and continued, in particular, in [Ba17], [PR17a], [Ba16b], [PR17b], [L+17], [BR17] and [EM17], on constructing e cient algorithms for computing (approximating) combinatori-ally de ned quantities (partition functions) by exploiting the information on their complex zeros. A typical application of the method consists of

a) proving that the function in question does not have zeros in some interesting domain in C^n

and

b) constructing a low-degree polynomial approximation for the logarithm of the function in a slightly smaller domain.

Usually, part a) is where the main work is done: since there is no general method to establish that a multivariate polynomial (typically having many monomials) is non-zero in a domain in C^n, some quite clever arguments are being sought and found, cf. [PR17b], [EM17], see also Section 2.5 of [Ba16b] for the very few general results in this respect. Once part a) is accomplished, part b) produces a quasi-polynomial approximation algorithm in a quite straightforward way, see Section 2.2 of [Ba16b]. However, if one wants to improve the complexity from quasi-polynomial to genuine polynomial time, a considerable e ort can be required, see [PR17a] and [L+17].

In this paper, we contribute a new method to accomplish part a) and demon-strate it by producing a quasi-polynomial algorithm to approximate permanents of complex matrices and tensors from a reasonably wide and interesting class (the hard work of sharpening our algorithms to genuine polynomial time under further restrictions on the matrices and tensors is done in [PR17a], see also [BR17]). While computing permanents of complex matrices is of interest to quantum computations and boson sampling, see [EM17], our main contribution is to the computation of multi-dimensional permanents of (complex) tensors, which results in an e cient algorithm to count perfect matchings, weighted by their Hamming distance to one given perfect matching, in an arbitrary hypergraph. In general, the algorithm is quasi-polynomial, it becomes genuinely polynomial on hypergraphs with degrees of the vertices bounded in advance. We discuss this in Section 1.7. We hope that the method of this paper will nd other applications, in particular to count solutions, weighted by their Hamming distance to one given solution, of other NP-complete problems.

We start with a popular example of the permanent of a square matrix.
(1.1) Permanents. Let A = (a_{ij}) be n n complex matrix and let

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

be its permanent. Here S_n is the symmetric group of all n! permutations of the set f1; : : : ; ng. First, we prove the following result.

(1.2) Theorem. Let A be n n complex matrix such that

n
X
$|a_{ij}| < 1$ for $i = 1; : : : ; n$:
j=1

Then

$per(I + A) \neq 0$;

where I is the n   n identity matrix.

In [Br59], Brenner obtains a family of inequalities satis ed by (determinantal) minors of a diagonally dominant matrix, mentions as a corollary (Corollary 5 from [Br59]), that the determinant of such a matrix is necessarily non-zero, and con-cludes the paper with the following sentence: \The referee remarked that since the permanent of matrix can be expanded by minors, corresponding theorems holds for permanents." The permanental version of Corollary 5 of [Br59] is equivalent to our Theorem 1.2.

In this paper, we provide a di erent proof of Theorem 1.2, which easily extends to multi-dimensional permanents of tensors (Theorem 1.5 below) and, more, gen-erally, can be useful for establishing a zero-free region for an arbitrary multi-a ne polynomial.

We also note that A is Hermitian then I +A, being a diagonally dominant matrix, is necessarily positive de nite and hence per(I + A) is positive real [MN62].

Let A be a matrix satisfying the conditions of Theorem 1.2. Then we can choose a continuous branch of the function A 7! ln per(I + A). Applying the methods developed in [Ba16b] and [Ba17], we obtain the following result.

(1.3) Theorem. Let us x a real $0 \ll 1$. Then for any positive integer n and for any $0 \ll 1$ there exists a polynomial $p(A) = p_{;n;}(A)$ in the entries of an
n -n complex matrix A such that $\deg p = O(\ln n \quad \ln)$ and

$|\ln per(I + A) \quad p(A)|$

provided $A = (a_{ij})$ is a complex n   n matrix such that

n
X
$|a_{ij}| <$ for $i = 1; : : : ; n$:
j=1

Moreover, given ; n and , the polynomial $p_{;n;}$ can be computed in $n^{O(\ln\ln)}$ (the implied constant in the \O" notation depends only on).

We note that the value of $| per(I + A)|$ for a matrix A satisfying the conditions of Theorem 1.3 can vary in an exponentially wide range in n, even when A is required to have zero diagonal: choosing A to be block-diagonal with 2 2 blocks

$$\begin{matrix} 0 & a \\ b & 0 \end{matrix} \;$$

we can make $| per(I + A)|$ as large as $(1 + {}^2)^{bn=2c}$ and as small as $(1 {}^2)^{bn=2c}$. If $B = (b_{ij})$ is a strongly diagonally dominant complex matrix such that

X $|b_{ij}|$

$$\sum_{j:\, j\neq i} |b_{ij}| \qquad\qquad \text{for} \qquad i = 1, \ldots, n$$

then

$$\text{per } B = \prod_{i=1}^{n} b_{ii} \ \text{per } C,$$

where $C = (c_{ij})$ is obtained from B by a row scaling

$$c_{ij} = b_{ii}^{-1} b_{ij} \qquad\qquad \text{for all} \quad i, j.$$

We have $C = I + A$, where A satisfies the conditions of Theorem 1.3. Hence our algorithm can be applied to approximate the permanents of complex strongly diagonally dominant matrices.

If the matrix A in Theorem 1.3 is Hermitian, then $I + A$, being diagonally domi-nant, is positive definite. A polynomial time algorithm approximating permanents of $n \times n$ positive semidefinite matrices within a simply exponential factor of c^n (with $c \approx 4.84$) is constructed in [A+17]. Other interesting classes of complex matrices where efficient permanent approximation algorithms are known are some random matrices [EM17] and matrices not very far from the matrix filled with 1s [Ba17]. Famously, there is a randomized polynomial time algorithm to approximate per A if A is non-negative real [J+04]. The best known deterministic polynomial time algorithm approximates the permanent of an $n \times n$ non-negative real matrix within an exponential factor of 2^n and is conjectured to approximate it within a factor of $2^{n/2}$ [GS14].

(1.4) Multidimensional permanents. For $d \geq 2$, let $A = (a_{i_1 \ldots i_d})$ be a cubical $n \times \cdots \times n$ array (tensor) of complex numbers (so that for $d = 2$ we obtain an $n \times n$ square matrix). We define the permanent of A by

$$\text{PER } A = \sum_{\sigma_2, \ldots, \sigma_d \in S_n} \prod_{i=1}^{n} a_{i\, \sigma_2(i) \ldots \sigma_d(i)}.$$

Clearly, our definition agrees with that of Section 1.1 for the permanent of a matrix. Just as the permanent of a matrix counts perfect matchings in the underlying weighted bipartite graph, the permanent of a tensor counts perfect matchings in the underlying weighted d-partite hypergraph, see, for example, Section 4.4 of [Ba16b]. We define the diagonal of a tensor A as the set of entries $\{a_{i \ldots i} : i = 1, \ldots, n\}$.
We prove the following extension of Theorem 1.2.

(1.5) Theorem. Let $A = (a_{i_1 \ldots i_d})$ be a d-dimensional $n \times \cdots \times n$ complex tensor with such that

$$\sum_{1 \leq i_2, \ldots, i_d \leq n} |a_{i\, i_2 \ldots i_d}| < 1 \quad \text{for} \quad i = 1, \ldots, n.$$

Then

$$\text{PER}(I + A) \neq 0,$$

where I is the d-dimensional $n \times \cdots \times n$ tensor with diagonal entries equal to 1 and all other entries equal to 0.

Similarly to Theorem 1.3, we deduce from Theorem 1.5 the following result:

(1.6) Theorem. Let us fix a positive integer $d \geq 2$ and a real $0 < \delta < 1$. Then for any positive integer n and any $0 < \epsilon < 1$ there exists a polynomial $p(A) = p_{d, \delta, n, \epsilon}(A)$

in the entries of a d-dimensional n : : : n tensor A such that deg p = O (ln n ln) and

$$ j\ln PER(I + A) \qquad p(A)j $$

for any d-dimensional n : : : n tensor A such that

1 i_2 X
                            d
              $ja_{ii_2:::i_d}j<$ for   i = 1; : : : ; n:
; : : : ; i   n

Moreover, given d, , n and , the polynomial $p_{d;\ ;n;}$ can be computed in $n^O d;^{(lnnln\ )}$ time (so that the implied constant in the \O" notation depends
only on d and   ).

(1.7) Weighted counting of perfect matchings in hypergraphs. We de-scribe an application of Theorem 1.6 to weighted counting of perfect matchings in hypergraphs, cf. [BR17]. Let H be a d-partite hypergraph with set V of nd vertices equally split among d pairwise disjoint parts $V_1; : : : ; V_d$ such that $V = V_1 [: : : [V_d$. The set E of edges of H consists of some d-subsets of V containing exactly one vertex from each part V_i. We number the vertices in each part V_i by 1; : : : ; n and encode H by a d-dimensional tensor A = ($a_{i1:::id}$), where

1   if fi_1; : : : ; i_dg 2 E;
$a_{i1:::i}d =$
          0   otherwise.

A perfect matching in H is a collection of n edges containing each vertex exactly once. As is known, for d 3 it is an NP-complete problem to determine whether a given d-partite hypergraph contains a perfect matching. Suppose, however, that we are given one perfect matching M_0 in H. Without loss of generality we assume that M_0 consists of the edges (1; : : : ; 1), (2; : : : ; 2); : : : ; (n; : : : ; n). For a perfect matching M in H, let dist(M; M_0) be the number of edges in which M and M_0 di er (the Hamming distance between M and M_0). Let us choose a > 0. It is not hard to see that

X

(1.7.1)   PER I+ ^{2} (A I)= $^{dist(M;M_0)}$;
M

where the sum is taken over all perfect matchings M in H. Let us assume now that each vertex of the part V_1 is contained in at most edges of H. It follows from Theorem 1.6 that (1.7.1) can be e ciently approximated for any , xed in advance, provided
r
                                                ———
                                                ———
< $^1_1$:

This is an improvement compared to [BR17] where we could only a ord = O(1= d) and also required the degree of every vertex of H not to exceed .

As is discussed in [BR17], see also [PR17a], for any , xed in advance, we obtain a polynomial time algorithm approximating (1.7.1).

Generally, knowing one solution in an NP-complete problem does not help one to nd out if there are other solutions. Our result shows that some statistics over the set of all solutions can still be computed e ciently.

We prove Theorems 1.2 and 1.5 in Section 2 and deduce Theorems 1.3 and 1.6 from them in Section 3.

2. Proofs of Theorems 1.2 and 1.5

We start with a simple lemma.

(2.1) Lemma. Let us x $_1$; : : : ; $_n$2C. Then for anyz_1; : : : ; z_n2Cthere exist
z ; : : : ; z 2 C such that
1   n

$$\sum_{k=1}^{n} z_k = \sum_{k=1}^{n} z_k ; \qquad \sum_{k=1}^{n} |z_k||z_k|$$

and $z_k \neq 0$ for at most one k.

Proof. Without loss of generality, we assume that $\neq 0$ for $k = 1, \ldots, n$. Given $z_1, \ldots, z_n$, let us define

$$K = \left\{(x_1, \ldots, x_n) \in \mathbb{C}^n : \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} z_k \text{ and } \sum_{k=1}^{n} |x_k| \leq \sum_{k=1}^{n} |z_k| \right\}$$

Then K is a non-empty compact set and the continuous function

$$(x_1, \ldots, x_n) \mapsto \sum_{k=1}^{n} |x_k|$$

attains its minimum on K at some point, say $(y_1, \ldots, y_n)$. We claim that all non-zero complex numbers among y_k are positive real multiples of each other.

Suppose that, say $y_1 \neq 0$ and $y_2 \neq 0$ are not positive real multiples of each other and let $v = y_1 + y_2$. Hence $|y_1| + |y_2| > |v|$. Let

$$y_1^0 = \frac{v|y_1|}{(|y_1| + |y_2|)} \text{ and } y_2^0 =$$

Then
$$y_1^0 + y_2^0 = v = y_1$$
$$\frac{v|y_2|}{(|y_1| + |y_2|)} :$$

$$+ \quad y_2$$

and

$$_y0 \ + \ _y0 \ = \ \frac{jvjjy_1j}{j\ _1y_1j + j\ _2y_2j} \ + \ \frac{jvjjy_2j}{j\ _1y_1j + j\ _2y_2j} \ < \ y \ + y \ :$$

$$j_1j \quad j \quad _2j \qquad\qquad\qquad 6 \qquad\qquad\qquad j \quad _1{}^j j \quad _2{}^j$$

Hence de ning $y_k{}^0 = y_k$ for $n > 2$, we obtain a point $(y_1{}^0; : : : ; y_n{}^0)$ 2 K with

$$\sum_{k=1}^{n} jy_k{}^0j < \sum_{k=1}^{n} jy_kj ;$$

which is a contradiction.

This proves that there is a point $(y_1; : : : ; y_n)$ 2 K where all non-zero complex numbers $_ky_k$ are positive real multiples of each other, so that

the ratios $\qquad \dfrac{_ky_k}{j\ _ky_kj} \qquad$ when $_ky_k \ 6= 0$ are all equal:

Next, we successively reduce the number of non-zero coordinates among $y_1; : : : ; y_n$, while keeping all non-zero numbers $_ky_k$ positive real multiples of each other.

Suppose that there are two non-zero coordinates, say y_1 and y_2. Without loss of generality, we assume that $j\ _1j \ j\ _2j$. Now, we let:

$$y_1{}^0 \ = 0 \quad \text{and} \quad y_2{}^0 \ = y_2 \ + \ \frac{j\ _1jjy_1j}{j\ _2jjy_2j} \ y_2:$$

Then

$$y^0 \ + \ y^0 \ = \ y^0 \ = \ y \ + \ j\ _1jjy_1j \qquad y \ + y \ :$$

$$j \qquad _1j \quad j\ _2j \qquad j\ _2j \qquad j\ _2j \qquad \overline{j\ _2j} \qquad j \qquad _1j\ j\ _2j$$

Moreover,

$$y^0 \qquad + \qquad _y0 = y \qquad + \qquad \frac{j\ _1jjy_1j}{_2j\ _2jjy_2j}y \ = \ y \ + \ \frac{_2y_2}{j\ _2y_2j^{j1}} \qquad y \ = \ y \ + \ \frac{_1y_1}{j\ _1y_1j^{j}} \qquad y$$

$$1\ _1 \quad 2 \qquad 2\ 2 \qquad 2 \qquad\qquad\qquad 2 \quad 2 \qquad\qquad\qquad 1j\ 2 \qquad 2 \qquad\qquad\qquad 1 \quad 1j$$

$$= \ _2y_2 + \ _1y_1:$$

Hence letting $y_k{}^0 = y_k$ for $k > 2$ we obtain a point $(y_1{}^0; : : : ; y_n{}^0)$ 2 K with fewer non-zero coordinates. Moreover, all non-zero numbers $_ky_k{}^0$ remain positive real

multiples of each other. Repeating this process, we obtain the desired vector

$(z_1 ; : : : ; z_n)$ 2 C

Now we are ready to prove Theorem 1.2.

(2.2) Proof of Theorem 1.2. First, we observe that without loss of generality, we may assume that A has zero diagonal. Indeed, let $A = (a_{ij})$ be an n n complex matrix with sums of the absolute values of entries in each row less than 1. Then the diagonal entry in the i-th row of the matrix $I + A$ is $1 + a_{ii}$ with absolute value $j1 + a_{ii}j$ $1 \ j \ a_{ii}j > 0$, while the sum of the absolute values of the o -diagonal entries in the i-th row of $I + A$ is less than $1j$ $a_{ii}j$. Consequently, dividing the i-row of $I + A$ by $1 + a_{ii}$ for $i = 1; : : : ; n$, we obtain the matrix $I + A^0$ where A^0 satis es the conditions of the theorem and, additionally, has zero diagonal. Moreover,

!

$$\text{per}(I + A) = \prod_{i=1}^{n} (1 + a_{ii}) \ \text{per}(I + A^0):$$

Thus we assume that A has zero diagonal.

Let M_n be the set of n n complex matrices A with zero diagonal and sums of absolute values of entries in every row less than 1. We claim for every A 2 M_n there exists B 2 M_n such that per(I + A) = per(I + B) and B has at most one non-zero entry in every row. Given A = (a_{ij}), we construct the matrix B step by step by modifying A row by row in n steps. Let A_{ij} be the (n 1) (n 1) matrix obtained from A by crossing the i-th row and the j-th column. We have

$$\text{per}(I + A) = \text{per} A_{11} + \sum_{j=2}^{n} a_{1j} \ \text{per} A_{1j}:$$

Applying Lemma 2.1, we nd b_{1j} for j = 1; : : : ; n such that $b_{11} = 0$,

$$\sum_{j=2}^{n} b_{1j} \ \text{per} A_{1j} = \sum_{j=2}^{n} a_{1j} \ \text{per} A_{1j}; \qquad \sum_{=2}^{n} j b_{1j} j < 1$$

and at most one of the numbers b_{1j} is non-zero. At the rst step, we de ne B by replacing a_{1j} by b_{1j} for j = 1; : : : ; n and note that per(I + A) = per(I + B).

At the end of the (k 1)-st step, we have a matrix B 2 M_n such that per(I+A) = per(I +B) and each of the rst k 1 rows of B contains at most one non-zero entry. If k n, we write

$$\text{per}(I + B) = \text{per} B_{kk} +$$

akj per Bkj;
j: j6=k

where B_{kj} is the (n 1) (n 1) matrix obtained from B by crossing out the k-th row and j-th column. Applying Lemma 2.1, we nd b_{kj} for j = 1; : : : ; n such that

$b_{kk} = 0$,

$$\sum_{j: j \neq k} b_{kj} \ \text{per} B_{kj} = \sum_{j: j \neq k} a_{kj} \ \text{per} B_{kj}; \qquad \sum_{j: j \neq k} j b_{kj} j < 1$$

and at most one of the numbers b_{kj} is non-zero. We modify B by replacing a_{kj} with b_{kj} for j = 1; : : : ; n. We have per(I + A) = per(I + B).

At the end of the n-th step, we obtain a matrix B 2 M_n containing at most one non-zero entry in each row and such that per(I + A) = per(I + B).

We now have to prove that per(I + B) 6= 0. Since every row of B contains at most one non-zero entry, the total number of non-zero entries in B is at most n. Therefore, if there is a column of B containing more than one non-zero entry, there is a column, say the k-th, lled by zeros only. Then per(I + B) = per(I + B^0), where B^0 is the matrix obtained from B by crossing out the k-th row and column. Hence without loss of generality, we may assume that every row and every column of B contains exactly one non-zero entry. Let us consider the bipartite graph on n + n vertices where the i-th vertex on one side is connected by an edge to the j-th 8

vertex on the other side if and only if the (i; j)-th entry of I + B is not zero. Then the graph is a disjoint union of some even cycles $C_1; : : : ; C_m$. Hence

$$
per(I + B) = \quad m \quad 0 \quad 1 + \quad {}^b_{ij} \quad 1 \\
Y \quad B \quad f \quad Y \quad C \\
B \quad i6{=}j \quad C \\
@ \quad\quad\quad A \\
i;jg2C_k \\
k{=}1
$$

where b_{ij} are the non-zero entries of B corresponding to the edges in C_k. Since $jb_{ij}j < 1$, we have per(I + B) 6= 0. Before we prove Theorem 1.5, we introduce a convenient de nition.

(2.3) Slice expansion of the permanent of a tensor. Let A = ($a_{i1:::id}$) be a d-dimensional n : : : n tensor. Let us x 1 k d and 1 j n. We de ne the (k; j)-th slice of A as the set of n^{d1} entries $a_{i1::: ;id}$ with $i_k = j$. In particular, if d = 2, so A is a matrix, the (k; j)-th slice of A is the j-th row if k = 1 and the j-th column if k = 2. Theorem 1.5 asserts that PER(I + A) 6= 0, if A is a complex tensor with sums of the absolute values of the entries in the (1; j)-th slice less than

1   for j = 1; : : : ; n.

For a given entry $a_{i1:::id}$, let $A_{i1:::id}$ be the d-dimensional (n 1) : : : (n 1) tensor obtained from A by crossing out the d slices containing $a_{i1:::id}$. Then, for

any k   = 1; : : : ; d and any j = 1; : : : ; n, we have the (k; j)-slice expansion of the

permanent:

$$
\begin{aligned}
&i_1 &&X \quad d \\
(2.3.1) \quad PER A &= &&{}^a_{i_1} \quad :::i_d \; {}^{PER A}_{i_1:::i_d} : \\
& &&;:::\; ;i \; : \\
&i_k{=}j
\end{aligned}
$$

(2.4) Proof of Theorem 1.5. The proof closely follows that of Theorem 1.2 in Section 2.2.

Arguing as in Section 2.2, without loss of generality we assume that A has zero diagonal.

Now we prove that for every d-dimensional n : : : n tensor A with zero diagonal and sums of absolute values in the (1; j)-th slice less than 1 for j = 1; : : : ; n, there exists a d-dimensional n : : : n tensor B with zero diagonal such that PER(I + A) = PER(I + B), all entries of B are less than 1 in the absolute value and the (1; j)-th slice of B contains at most one non-zero entry for j = 1; : : : ; n.

We construct B by modifying A slice by slice starting with B = A. If the (1; j)-th slices for j = 1; : : : ; m 1 are modi ed, we consider the (1; m)-slice expansion (2.3.1)

$$
\begin{aligned}
&X \\
PER(I{+}B) = PER B&{}_{m:::m} + \quad {}^a_{mi_2:::i_d} {}^{PER B}_{mi_2:::i_d} : \\
&i_2;::: \;;i_d: \\
&(i_2;::: \;;i_d)6{=}(m;::: \;;m)
\end{aligned}
$$

Using Lemma 2.1, we nd complex numbers $b_{mi2:::id}$ such that $b_{m:::m} = 0$, at most one of $b_{mi2:::id}$ is non-zero, less than 1 in the absolute value and

$$
\begin{aligned}
2 \quad {}^X_d &&&&& i_2 \quad {}^X_d \\
{}^a_{mi_2} \quad :::i_d \; {}^{PER B}_{mi_2:::i_d} &= &&&&{}^b_{mi_2} \quad :::i_d \; {}^{PER B}_{mi_2:::i_d} : \\
i \quad ;:::\;;i \; : &&&&& ;:::\;;i \; : \\
(i_2;:::\;;i_d)6{=}(m;:::\;;m) &&&&& (i_2;:::\;;i_d)6{=}(m;:::\;;m)
\end{aligned}
$$

We then replace the entries $a_{mi2\dots id}$ in the $(1; m)$-th slice of B by the entries $b_{mi2\dots id}$.
In the end we produce the desired tensor B.

Hence our goal is to prove that PER(I +B) $\neq$ 0 if B is a d-dimensional n : : : n tensor with zero diagonal, complex entries less than 1 in the absolute value and containing at most one non-zero entry in the $(1; j)$-th slice for $j = 1; : : : ; n$. In particular, the total number of non-zero entries of B does not exceed n. It follows then that if some $(k; j)$-th slice of B contains more than one non-zero entry, there is a $(k; m)$-th slice of B lled entirely by zeros. Denoting by B^0 the $(n\ 1) : : : (n\ 1)$ tensor obtained from B by crossing out the d slices containing $(m; : : : ; m)$, we will have PER(I + B) = PER(I + B^0). Hence without loss of generality we assume that B contains exactly one non-zero entry in each of the dn slices.

Let us consider the underlying d-partite hypergraph H encoded by I + B. We number the vertices in each of the d parts by $1; : : : ; n$. A set E of d vertices of

H   is an edge of H if and only if it contains exactly one vertex i_k for $k = 1; : : : ; d$ from each part and the $(i_1; : : : ; i_d)$-th entry of I + B is non-zero. Hence each vertex is contained in exactly two edges of H. We call the edges $e_j = (j; : : : ; j)$ for $j = 1; : : : ; n$ standard (they correspond to the diagonal 1s of I + B) and the remaining n edges additional (they correspond to the non-zero entries of B). We de ne the weight of a standard edge equal to 1 and of an additional edge f$i_1; : : : ; i_d$g

equal to $b_{i1\dots id}$. We de ne the weight of a perfect matching M in H as the product of weights of the edges in M and the total weight w(H) as the sum of weights of all perfect matchings in H. Hence PER(I + B) = w(H) and our goal is to prove that w(H) $\neq$ 0.
The hypergraph H splits into the disjoint union of connected components
$H_1; : : : ; H_m$, so that for any two vertices v^0 and v^{00} in each H_k there is a chain of
edges u ; : : : ; u   such that v^0 u , v^{00} u   and u   u =   for i = 1; : : : ; s 1
1   s   2 1   2   s   i\   i+1 6 ;
and for any two vertices v^0 and v^{00} in di erent H_i and H_j there is no such chain. Then w(H) = w(H$_1$) w(H$_k$) and hence it su ces to prove that w(H$_k$) $\neq$ 0 for $k = 1; : : : ; m$. Hence without loss of generality, we assume that H is connected. There is a perfect matching of weight 1 in H consisting of the standard edges $e_1; : : : ; e_n$, which we call standard. Furthermore, n additional edges are necessarily pairwise disjoint (since otherwise there is a vertex that belongs to at least three edges: two additional and one standard) and also form a perfect matching, which we call additional.

We claim that any perfect matching in H is either standard or additional. Seek-ing a contradiction, suppose that there is a perfect matching consisting of some 1 k < n standard edges, say $e_1; : : : ; e_k$ and some n k > 0 additional edges

edges, say $u_{k+1}; : : : ; u_n$. Let U = u_{k+1} [: : : [u_n and let W = e_1 [: : : [e_k. Then

every vertex in U is contained in one standard edge among $e_{k+1}; : : : ; e_n$ and one additional edge among $u_{k+1}; : : : ; u_n$ and hence there cannot be an edge contain-ing a vertex of U and a vertex of W , which contradicts the assumption that H is connected.

Hence there are exactly two perfect matchings in connected H, one standard of weight 1 and one additional of weight less than 1 in the absolute value. Thus

w(H) $\neq$ 0 and the proof follows.

3. Proofs of Theorems 1.3 and 1.6

We follow the approach outlined in [Ba16b] and [Ba17].

(3.1) Proof of Theorem 1.3. Let A be an n n complex matrix satisfying the conditions of Theorem 1.3. We consider a univariate polynomial

g(z) = per(I + zA):

Then deg g   n and, by Theorem 1.2, we have

(3.1.1) $\qquad g(z) \neq 0$ provided $\qquad |z| < 1:$

We choose a continuous branch of

(3.1.2) $\qquad f(z) = \ln g(z)$ for $\qquad |z| < 1:$

Hence our goal is to approximate $f(1) = \ln \text{per}(I + A)$ within an additive error of > 0 by a polynomial $p(A)$ in the entries of A of deg $p = O(\ln n \ln)$. We consider the Taylor polynomial $T_m(z)$ of f computed at $z = 0$:

$$(3.1.3) \quad T_m(z) = f(0) + \sum_{k=1}^{m} \frac{f^{(k)}(0)}{k!} z^k:$$

Since deg g n and (3.1.1) holds, by Lemma 2.2.1 of [Ba16b] (Lemma 7.1 of [Ba17]), we have

$$(3.1.4) \qquad |f(1) - T_m(1)| \qquad \frac{n}{(m+1)(1)}^{m+1}:$$

Therefore, to approximate $f(1)$ within an absolute error $0 \ll 1$, it suces to choose $m = O(\ln n \ln)$, where the implied constant in the "O" notation depends only on .

Next, as is discussed in Section 2.2.2 of [Ba16b] (Section 7.1 of [Ba17]), one can compute the values of $f(0) = \ln g(0) = 0$ and $f^{(k)}(0)$ for $k = 1; : : : ; m$ from

the values of $g(0) = 1$ and $g^{(k)}(0)$ for $k = 1; : : : ; m$ in $O(m^2)$ time, by solving a non-singular m m triangular system of linear equations

$$(3.1.5) \qquad \sum_{j=0}^{k-1} \binom{k-1}{j} f^{(k-j)}(0) g^{(j)}(0) = g^{(k)}(0) \quad \text{for } k = 1; : : : ; m:$$

On the other hand,

$$g(z) = \text{per}(I + zA) = \sum_{\substack{I \subseteq \{1;::::;n\}}} z^{|I|} \text{per } A_I$$

where A_I is the submatrix of A consisting of the entries with row and column indices in I, where we agree that per $A_; = 1$. Consequently,

$$g^{(k)}(0) = k! \sum_{\substack{I \subseteq \{1;::::;n\} \\ |I|=k}} \text{per } A_I ;$$

so that $g^{(k)}(0)$ is a homogeneous polynomial of degree k in the entries of A. It can be computed by the direct enumeration in $n^{O(k)}$ time. It follows from (3.1.5) that $f^k(0)$ is a polynomial of degree k in the entries of A computable in $n^{O(k)}$ time.

Since $m = O(\ln n \ln)$, the proof follows by (3.1.4).

(3.2) Proof of Theorem 1.6. The proof is very similar to that of Theorem 1.3 in Section 3.1. Let A be a d-dimensional n : : : n complex tensor satisfying the conditions of Theorem 1.6. We consider the univariate polynomial

$g(z) = PER(I + zA)$:

Then deg g n and by Theorem 1.5, we have (3.1.1). We then de ne the function f(z) by (3.1.2), de ne the polynomial $T_m(z)$ by (3.1.3) and conclude as in Section 3.1 that (3.1.4) holds. As in Section 3.1, to approximate f(1) by $T_m(1)$ within an additive error $0 \ll 1$, it su ces to choose m = O (ln n ln).

As in Section 3.1, it remains to prove that $g^{(k)}(0)$ is a polynomial of degree k in the entries of A. We have the expansion

I f^X g

$PER(I + zA) =$ $z^{jIj} PER A_I$;
1;::: ;n

where A_I is the subtensor k = 1; : : : ; d and where we

g

of A consisting of the entries $a_{i1:::id}$ with i_k 2 I for agree that PER A; = 1. Then
X
$^{(k)}(0) = k!$ PER A_I ;

I f1;::: ;ng
jIj=k

so $g^{(k)}(0)$ is indeed a homogeneous polynomial of degree k in the entries of A. Moreover, it can be computed by the direct enumeration in $n^{O(kd)}$ time. Since we
have k = O (ln n ln), the proof is completed as in Section 3.1.

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